Bernoulli's Differential Equation

Solve the following differential equation

$$(y + xy^2)dx - xdy = 0$$

Solution

Rearrange:

$$(y + xy^2) = xy'$$

$$\frac{y}{x} + y^2 = y'$$

Bernoulli's differential equation

$$y' - \frac{y}{x} = y^2$$

Divide everything by y^2

$$\frac{y'}{y^2} - \frac{y^{-1}}{x} = 1$$

Let $z = y^{-1}$, then $z' = -y^{-2}y'$, $-z' = y^{-2}y'$

$$-z' - \frac{z}{x} = 1$$

Make a substitution: $z = uv$, then $z' = u'v + uv'$

$$-u'v - uv' - \frac{uv}{x} = 1$$

$$-v(u' + u/x) - uv' = 1$$

Solve a system of equations, where $v(u' + u/x) = 0$ and $-uv' = 1$

$$u' + u/x = 0$$

$$du/dx = -u/x$$

$$du/u = -dx/x$$

$$\ln(u) = -\ln(x)$$

$$\ln(u) = \ln(x^{-1})$$

$$u = 1/x$$

Then with this we solve the second expression:

$$-\frac{1}{x}v' = 1$$

$$dv/dx = -x$$

$$dv = -xdx$$

$$v = -\frac{x^2}{2} + C$$

With this, I obtain $z = \frac{1}{x}(C - \frac{x^2}{2})$, so $z = \frac{C}{x} - \frac{x}{2}$ and with this I obtain $y = \frac{1}{\frac{C}{x} - \frac{x}{2}}$.

$$y = \frac{1}{\frac{2C - x^2}{2x}}$$

$$y = -\frac{2x}{K + x^2}$$